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A REALIZATION THEORY FOR AUTONOMOUS BOUNDARY-VALUE LINEAR SYSTEMS

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abstract A *frequency-domain* realization theory is developed for the class of autonomous, but not necessarily stationary, boundary-value linear systems. It is shown that this realization problem, which consists of constructing autonomous boundary-value linear systems from prescribed weighting patterns, reduces to the factorization of several rational matrices in two variables having separable denominators. This factorization problem is examined and a method is given for constructing minimal factorizations for such rational matrices. The special case of stationary systems is also considered.

Une théorie de la réalisation pour les systèmes linéaires aux deux bouts autonomes

Résumé Dans ce rapport, on développe une théorie de réalisation dans le domaine fréquentiel pour les systèmes linéaires aux deux bouts qui sont autonomes mais pas forcément stationnaires. La construction des systèmes autonomes aux deux bouts à partir de relations entrée-sortie se réduit à la factorisation de matrices rationnelles à deux variables ayant des dénominateurs séparables. On examine ce problème de factorisation et on propose une méthode pour la construction des facteurs minimaux. De plus, on considère le cas particulier des systèmes stationnaires.

1 Introduction

Boundary-value models arise naturally in describing physical phenomena where the independent variable is space rather than time and differ significantly from the classical initial-value models in that they are inherently non-causal. In recent years, there has been a large effort to develop deterministic and stochastic system theories for the class of linear boundary value systems. In the deterministic case, basic notions of classical linear system theory such as controllability, observability, minimality and irreducibility have been extended to the case of boundary value systems [1]-[4], [6], and in the stochastic case, the smoothing [16]-[18] and the stochastic realization problems [15] have been studied. A class of boundary-value systems which is of particular interest is the class of autonomous (constant-coefficient, time-invariant) systems:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad a \leq t \leq b, \tag{1}$$

$$V_i x(a) + V_f x(b) = 0, \tag{2}$$

$$y(t) = Cx(t), \qquad a \leq t \leq b, \tag{3}$$

where A, C and B are $n \times n$, $m \times n$ and $n \times p$ constant matrices respectively, and V_i and V_f are $n \times n$ boundary matrices. Results on autonomous boundary-value systems, and their discrete time counterparts, the two-point boundary-value descriptor systems [7], can be found in [2], [5], [9]-[14].

A powerful tool used in the study of the realization problem for classical, time-invariant linear systems is the Laplace transform (z-transform in the discrete case). In particular in that case, the application of the Laplace transform to the impulse response which is to be realized, reduces the realization problem into that of factoring a rational matrix [19], [20]. Our goal in this paper is to develop a similar technique for realizing autonomous boundary value systems from their *weighting patterns* (input-output maps). This problem has been considered in [13] for stationary systems (system having shift-invariant weighting patterns [2], [5], [10]) which constitute a sub-class of autonomous systems. In that case, the realization theory resembles more the classical causal realization theory because the weighting pattern is just a function of one variable. In general however, autonomous systems are not stationary i.e. their weighting patterns depend on two variables. For this reason, in this paper we use the 2D Laplace transformation and the transform domain realization theory that we develop based on this transformation involves rational matrices in two variables. In turns out that we are dealing only with rational matrices in two variables which have separable denominators [21]-[24].

The outline of this paper is as follows. In Section 2, we present some basic results on the well-posedness of boundary-value systems and establish the notation which will be used throughout the paper. We characterize the class of minimal boundary-value linear systems in Section 3. In Section 4, we present some results concerning factorization of rational matrices in two variables having separable denominators. We use these results in Section 5 to obtain necessary and sufficient conditions for realizability of candidate weighting patterns. We use the results of Section 4 also in Section 6 to compute the dimension of minimal realizations of a realizable weighting pattern and to construct its minimal realizations. The special case of stationary systems is considered in Section 7. For simplicity, we only consider continuous-time systems.

2 Preliminaries

In this paper, we are only concerned with well-posed systems, i.e. systems where the state $x(t)$ and consequently the output $y(t)$ are uniquely determined in terms of the input function $u(t)$ supposed to be square integrable. The boundary-value system (1-3) is well-posed if and only if [2]

$$\det(V_i \exp(aA) + V_f \exp(bA)) \neq 0. \tag{4}$$

Assuming (4) is satisfied, System (1-3) defines a well-defined input-output map

$$y(t) = \int_a^b w(t,\tau)u(\tau)d\tau \tag{5}$$

where $w(t,\tau)$ denotes the weighting pattern of the system given by

$$w(t,\tau) = \begin{cases} C\exp((t-a)A)(I-P)\exp(-(\tau-a)A)B, & a \leq \tau < t \leq b, \\ -C\exp((t-a)A)P\exp(-(\tau-a)A)B, & a \leq t < \tau \leq b, \end{cases} \tag{6}$$

and where P is the *canonical boundary-value operator* [3] given by

$$P = \exp(aA)\{(V_i \exp(aA) + V_f \exp(bA))^{-1} V_f \exp(bA)\} \exp(-aA). \tag{7}$$

Notice that the weighting pattern is completely defined in terms of the matrices C, P, A, and B. We call (C,P,A,B) a realization of the weighting pattern w; a boundary-value system having w for weighting pattern can easily be constructed from (C,P,A,B).[1] The problem considered in this paper then consists of constructing *minimal realizations* [2] given w.

The causal and the anti-causal parts of w are analytic functions and thus we can consider their analytical extensions to arbitrary intervals. Consider the extensions of the two parts of w to the interval $[a,\infty)$, and denote the causal and the anti-causal parts respectively by w_c and w_a:

$$\begin{aligned} w_c(t,\tau) &= C\exp((t-a)A)(I-P)\exp(-(\tau-a)A)B, \quad a \leq \tau, t, & (8)\\ w_a(t,\tau) &= -C\exp((t-a)A)P\exp(-(\tau-a)A)B, \quad a \leq \tau, t. & (9) \end{aligned}$$

We define the transforms of w_c and w_a as follows

$$\begin{aligned} W_c(s,\sigma) &= \int_a^\infty \int_a^\infty w_c(t,\tau)e^{-s(t-a)-\sigma(\tau-a)}dtd\tau \\ &= C(sI-A)^{-1}(I-P)(\sigma I + A)^{-1}B, & (10)\\ W_a(s,\sigma) &= \int_a^\infty \int_a^\infty w_a(t,\tau)e^{-s(t-a)-\sigma(\tau-a)}dtd\tau \\ &= -C(sI-A)^{-1}P(\sigma I + A)^{-1}B. & (11) \end{aligned}$$

[1] Boundary matrices V_i and V_f can be obtained as follows: $V_f = DP\exp((a-b)A)$ and $V_i = D(I-P)$ where D is any invertible matrix.

[2] (C,P,A,B) is a minimal realization of w if A has smallest dimension among all realizations of w.

W_c and W_a are essentially the 2D Laplace transforms of w_c and w_a. Constructing W_c and W_a form w is straightforward and thus we shall assume that they are available. The realization problem considered in this paper then becomes the following. Given rational matrices $W_c(s,\sigma)$ and $W_a(s,\sigma)$ find (C,P,A,B) such that (10) and (11) are satisfied and A has minimal dimension. Of course not all rational matrix pairs $\{W_c, W_a\}$ are realizable. In Section 5, we characterize the class of realizable rational matrix pairs and in Section 6, we present a method for constructing minimal realizations via factorization of rational matrices in two variables. But before we start, we need to characterize the class of minimal systems.

3 Minimal systems

In [5], necessary and sufficient conditions for minimality of autonomous boundary-value linear systems have been derived.

Theorem 3.1 ([5]) *System (1-3) is minimal if and only if*

$$\bigcap_{i,j=0}^{n-1} Ker \begin{bmatrix} CA^iP \\ CA^i(I-P) \end{bmatrix} A^j = 0, \tag{12}$$

$$\bigvee_{i,j=0}^{n-1} Im\, A^i \begin{bmatrix} PA^jB & (I-P)A^jB \end{bmatrix} = I\!R^n, \tag{13}$$

$$\bigcap_{i=0}^{n-1} Ker\, CA^i \subset \bigvee_{i=0}^{n-1} Im\, A^iB, \tag{14}$$

where P is the canonical boundary operator defined in (7).

Reachability and observability are not necessary for minimality, that is why minimal realizations are not necessarily similar; there are some additional degrees of freedom. We characterize exactly how minimal realizations are related in the next theorem and corollary. This result generalizes similar result obtained for stationary systems in [2].

Theorem 3.2 *Suppose (C_1, P_1, A_1, B_1) and (C_2, P_2, A_2, B_2) are minimal realizations of a weighting pattern w, then there exists an invertible matrix T such that*

$$\begin{aligned} B_2 &= TB_1, & (15)\\ C_2 &= C_1T^{-1}, & (16)\\ O_1(A_1 - T^{-1}A_2T) &= 0, & (17)\\ (A_1 - T^{-1}A_2T)R_1 &= 0, & (18) \end{aligned}$$

and

$$O_1(P_1 - T^{-1}P_2T)R_1 = 0, \tag{19}$$

where R_1 and O_1 denote respectively the reachability and observability matrices[3] *of System 1.*

[3]The reachability matrix of (A_i, B_i) is $R_i = \begin{bmatrix} B_i & A_iB_i & .. & A_i^{n-1}B_i \end{bmatrix}$ and the observability matrix of (C_i, A_i) is $O_i^T = \begin{bmatrix} C_i^T & A_i^TC_i^T & .. & (A_i^{n-1})^TC_i^T \end{bmatrix}$, $i = 1, 2$.

Proof Let for $i = 1,2$,

$$\Gamma_i = \left[\begin{array}{cc} (I-P_i)R_i & -P_iR_i \end{array} \right], \tag{20}$$

then by noting that

$$\begin{aligned} \frac{\partial^{j+k} w_c(t,\tau)}{\partial^j t \partial^k \tau}|_{t=\tau=a} &= (-1)^k C_1 A_1^j (I-P_1) A_1^k B_1 \\ &= (-1)^k C_2 A_2^j (I-P_2) A_2^k B_2, \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial^{j+k} w_a(t,\tau)}{\partial^j t \partial^k \tau}|_{t=\tau=a} &= -(-1)^k C_1 A_1^j P_1 A_1^k B_1 \\ &= -(-1)^k C_2 A_2^j P_2 A_2^k B_2, \end{aligned} \tag{22}$$

we deduce that for all $l \geq 0$,

$$C_1 A_1^l \Gamma_1 = C_2 A_2^l \Gamma_2. \tag{23}$$

This implies that

$$O_1 R_1^w = O_2 R_2^w, \tag{24}$$

where R_i^w, $i = 1,2$, denotes the reachability matrix of (A_i, Γ_i). Thanks to the second minimality condition, (A_1, Γ_1) and (A_2, Γ_2) are reachable and thus (24) implies that O_1 and O_2 have equal ranks. If we now let

$$U = R_2^w (R_1^w)^T (R_1^w (R_1^w)^T)^{-1}, \tag{25}$$

it follows immediately that

$$O_2 U = O_1. \tag{26}$$

We can find a similar identity for the reachability matrices. Let

$$\Omega_i = \left[\begin{array}{c} O_i(I-P_i) \\ -O_iP_i \end{array} \right], \quad i = 1,2, \tag{27}$$

then we can show that

$$O_1^w R_1 = O_2^w R_2, \tag{28}$$

where we have denoted the observability matrix of (Ω_i, A_i), $i = 1,2$, by O_i^w. Thanks to the minimality assumption and Theorem 3.1, we can show that (Ω_i, A_i), $i = 1,2$, is observable and thus from (28) follows that R_1 and R_2 have equal ranks. Moreover, we have that

$$R_2 = V R_1, \tag{29}$$

where

$$V = ((O_2^w)^T O_2^w)^{-1} (O_2^w)^T O_1^w. \tag{30}$$

In addition to the two *Hankel* matrices (24) and (28), there exists the usual Hankel matrix identity

$$O_1 R_1 = O_2 R_2. \tag{31}$$

This identity is obtained by noting that

$$\frac{\partial^k (w_c(t,\tau) - w_a(t,\tau))}{\partial^k t}|_{t=\tau=a} = C_1 A_1^k B_1 = C_2 A_2^k B_2. \tag{32}$$

Even though U as defined in (25) is not always invertible, thanks to (26) and the fact that O_1 and O_2 have equal ranks, we can find an invertible matrix $\hat{U}$ such that

$$O_2 \hat{U} = O_1. \tag{33}$$

Similarly, there exists an invertible matrix $\hat{V}$ such that

$$R_2 = \hat{V} R_1. \tag{34}$$

Notice that because of (31), $\hat{U}$ and $\hat{V}$ must also satisfy

$$O_2(\hat{U} - \hat{V}) R_1 = 0. \tag{35}$$

The question is then whether or not we can choose $\hat{U} = \hat{V}$. To see that this can be done, assume that we have chosen a basis for system i, $i = 1,2$, compatible with the direct sum decomposition $\mathcal{O}_i \oplus [\mathcal{O}_i^{\perp} \cap \mathcal{R}_i] \oplus \mathcal{R}_i$ where $\mathcal{O}_i$ and $\mathcal{R}_i$ denote the unobservability and the reachability subspaces of system i, respectively, i.e. $\mathcal{O}_i = \mathrm{Ker}\ O_i$ and $\mathcal{R}_i = \mathrm{Im}\ R_i$. Expression (34) implies that in these new bases $\hat{V}$ must have the following structure

$$\hat{V} = \begin{pmatrix} \hat{V}_1 & \hat{V}_2 & \star \\ \hat{V}_3 & \hat{V}_4 & \star \\ 0 & 0 & \star \end{pmatrix}, \tag{36}$$

where $\hat{V}_1$, $\hat{V}_2$, $\hat{V}_3$ and $\hat{V}_4$ are fixed and $\star$'s are arbitrary. Similarly, (33) implies that $\hat{U}$ must have the following structure

$$\hat{U} = \begin{pmatrix} \star & \star & \star \\ 0 & \hat{U}_1 & \hat{U}_2 \\ 0 & \hat{U}_3 & \hat{U}_4 \end{pmatrix}. \tag{37}$$

Because of (35), $\hat{U}$ and $\hat{V}$ are related as follows

$$\hat{U}_1 = \hat{V}_4, \quad \hat{V}_3 = \hat{U}_3 = 0. \tag{38}$$

Now if we let

$$T = \begin{pmatrix} \hat{V}_1 & \hat{V}_2 & \hat{U}_2 \\ 0 & \hat{V}_4 & \hat{U}_2 \\ 0 & 0 & \hat{U}_4 \end{pmatrix}, \tag{39}$$

then it is straightforward to verify that T is invertible (since $\hat{U}$ and $\hat{V}$ are invertible and block upper-triangular, $\hat{V}_1$, $\hat{V}_4$, and $\hat{U}_4$ are invertible) and

$$\begin{aligned} R_2 &= T R_1 && (40) \\ O_2 &= O_1 T^{-1}. && (41) \end{aligned}$$

Matrix T that we have constructed verifies (15-18) thanks to (40) and (41). It must also verify (18) because, thanks to (22) we have that

$$O_1P_1R_1 = O_2P_2R_2. \tag{42}$$

This completes the proof of Theorem 4.2. □

Corollary *Suppose (C_1, P_1, A_1, B_1) is a minimal realization of a weighting pattern w and T any invertible matrix. Then any (C_2, P_2, A_2, B_2) satisfying (15-19) is also a minimal realization of w.*

Proof Let w_i, $i = 1,2$, represent the weighting pattern of (C_i, P_i, A_i, B_i). Then what we need to show is that $w_1 = w_2$. Note that since C_2, P_2, A_2, and B_2 satisfy (15-18) , there exist matrices J and K satisfying

$$O_1J = KR_1 = 0, \tag{43}$$

and a matrix L satisfying

$$O_1L = LR_1 = 0 \tag{44}$$

such that

$$B_2 = TB_1, \tag{45}$$
$$C_2 = C_1T^{-1}, \tag{46}$$
$$A_2 = T(A_1 + L)T^{-1}, \tag{47}$$
$$P_2 = T(P_1 + J + K)T^{-1}. \tag{48}$$

Now let $w_{c,i}$, $i = 1,2$, denote the causal part of w_i, and suppose for simplicity that $a = 0$. Then

$$\begin{aligned} w_{c,2}(t,\tau) &= C_2exp(tA_2)(I - P_2)exp(-\tau A_2)B_2 \\ &= C_1T^{-1}exp(tT(A_1 + L)T^{-1})(I - T(P_1 + J + K)T^{-1})exp(-\tau T(A_1 + L)T^{-1})TB_1 \\ &= C_1exp(t(A_1 + L))(I - (P_1 + J + K))exp(-\tau(A_1 + L))B_1. \end{aligned} \tag{49}$$

But because of (44)

$$\begin{aligned} C_1exp(t(A_1 + L)) &= C_1(I + tA_1 + tL + \frac{t^2A_1^2}{2} + \frac{t^2A_1L}{2} + \frac{t^2LA_1}{2} + \frac{t^2L^2}{2} + \ldots) \\ &= C_1 + tC_1A_1 + \frac{t^2C_1A_1^2}{2} + \ldots \\ &= C_1exp(tA_1), \end{aligned} \tag{50}$$

and

$$
\begin{aligned}
exp(-\tau(A_1+L))B_1 &= (I-\tau A_1-\tau L+\frac{\tau^2A_1^2}{2}+\frac{\tau^2A_1L}{2}+\frac{\tau^2LA_1}{2}+\frac{\tau^2L^2}{2}+...)B_1 \\
&= B_1-\tau A_1B_1+\frac{\tau^2A_1^2B_1}{2}+... \\
&= exp(-\tau A_1)B_1. \qquad (51)
\end{aligned}
$$

Thus

$$w_{c,2}=C_1exp(tA_1)(I-(P_1+J+K))exp(-\tau A_1)B_1, \qquad (52)$$

which since $C_1exp(tA_1)(J+K)exp(-\tau A_1)B_1=0$ (thanks to (43)), implies that

$$w_{c,2}=C_1exp(tA_1)(I-P_1)exp(-\tau A_1)B_1=w_{c,1}. \qquad (53)$$

We can similarly show that $w_{a,2}=w_{a,1}$. □

Theorem 3.1, Theorem 3.2 and its corollary completely characterize the class of minimal realizations. Note that for the special case of causal systems, from Theorem 3.1, we obtain the classical conditions for minimality, i.e. reachability and observability, and from Theorem 3.2, we get that all minimal systems are similar, which of course is a well-known result.

So far, our analysis has been restricted to the time-domain. From this point on however we shall be more concerned with the transform-domain (frequency-domain). The transform domain analysis in our case involves a special class of rational matrices in two variables, in particular rational matrices in two variables which have separable denominators (in the causal case, only rational matrices in one variable are involved). In the next section, we present some results concerning factorization of rational matrices in two variables having separable denominators. These results are essential for the development of our realization theory.

4 Transform domain

The goal of this paper is to develop a transform domain realization theory for boundary-value systems. The transformation which we use for this theory is the 2D Laplace transform introduced in Section 2. It is straightforward to see from (10) and (11) that we are dealing with rational matrices in two variables which are strictly proper and have separable denominators. A rational matrix $H(s,\sigma)$ is called strictly proper if

$$\lim_{s\to\infty}H(s,\sigma)=\lim_{\sigma\to\infty}H(s,\sigma)=0; \qquad (54)$$

it has separable denominator if it can be expressed as

$$H(s,\sigma)=\frac{N(s,\sigma)}{p(s)q(\sigma)}, \qquad (55)$$

where N is a polynomial matrix in two variable, and p and q are saclar polynomials.

Theorem 4.1 *Let $H(s,\sigma)$ be a strictly proper rational matrix having separable denominator. Then there exist strictly proper rational matrices $L(s)$ and $R(\sigma)$ such that*

$$H(s,\sigma) = L(s)R(\sigma). \tag{56}$$

Proof We shall start the proof by first showing the following lemma:

Lemma 4.1 *Let $N(s,\sigma)$ be a polynomial matrix in s and σ. Then there exist polynomial matrices $F(s)$ and $G(\sigma)$ such that*

$$N(s,\sigma) = F(s)G(\sigma). \tag{57}$$

Proof Matrix N is polynomial in s and σ, i.e. there exist constants $[M_{ij}]_{kl}$ such that

$$N_{ij}(s,\sigma) = \sum_{k=0}^{\alpha_{ij}} \sum_{l=0}^{\beta_{ij}} [M_{ij}]_{kl} s^k \sigma^l \tag{58}$$

where α_{ij} and β_{ij} are the degrees of the polynomial H_{ij} (the ij^{th}-entry of H) in s and σ respectively. By adding obvious zero terms to (58), we can rewrite it as

$$N_{ij}(s,\sigma) = \sum_{k=0}^{\alpha_i} \sum_{l=0}^{\beta_j} [M_{ij}]_{kl} s^k \sigma^l \tag{59}$$

where $\alpha_i = \max_j \alpha_{ij}$ and $\beta_j = \max_i \beta_{ij}$. From (59) follows immediately that

$$H(s,\sigma) = S(s) M \Sigma(\sigma) \tag{60}$$

where M is a constant matrix made of block entries M_{ij} defined in (59) and

$$S(s) = \begin{pmatrix} 1 & s & .. & s^{\alpha_1} & & & & & & & & & \\ & & & & 1 & s & .. & s^{\alpha_2} & & & & & \\ & & & & & & & & \cdots & & & & \\ & & & & & & & & & 1 & s & .. & s^{\alpha_r} \end{pmatrix}, \tag{61}$$

$$\Sigma(\sigma) = \begin{pmatrix} 1 & & & \\ \sigma & & & \\ \vdots & & & \\ \sigma^{\beta_1} & & & \\ & 1 & & \\ & \sigma & & \\ & \vdots & & \\ & \sigma^{\beta_2} & & \\ & & \vdots & \\ & & & 1 \\ & & & \sigma \\ & & & \vdots \\ & & & \sigma^{\beta_c} \end{pmatrix}, \tag{62}$$

and where r and c denote the number of rows and columns of H respectively. The Lemma is now proved because we can choose F and G as follows

$$F(s) = S(s)M, \quad G(\sigma) = \Sigma(\sigma), \tag{63}$$

or the other way around:

$$F(s) = S(s), \quad G(\sigma) = M\Sigma(\sigma). \tag{64}$$

We could also factor M as $M = UV$ in which case we can have

$$F(s) = S(s)U, \quad G(\sigma) = V\Sigma(\sigma). \tag{65}$$

This completes the proof of Lemma 3.1. □

Continuing with the proof of the theorem, note that we can express $H(s,\sigma)$ as follows

$$H(s,\sigma) = \frac{N(s,\sigma)}{p(s)q(\sigma)} \tag{66}$$

where N is a polynomial matrix, and, p and q are polynomial scalars. Moreover, the degree in s of $N(s,\sigma)$ is strictly smaller than that of $p(s)$, similarly, the degree in σ of $N(s,\sigma)$ is strictly smaller than that of $q(\sigma)$. From the previous lemma, we know that we can factor $N(s,\sigma)$ as follows

$$N(s,\sigma) = F(s)G(\sigma). \tag{67}$$

In particular, we can choose F and G such that the degree of $F(s)$ equals the degree of $N(s,\sigma)$ in s and the degree of $G(\sigma)$ equals the degree of $N(s,\sigma)$ in σ (all the F and G's constructed in the proof of the lemma have these properties). It is clear now that we can pick

$$\begin{aligned} L(s) &= \frac{F(s)}{p(s)} \\ R(\sigma) &= \frac{G(\sigma)}{q(\sigma)}. \end{aligned} \tag{68}$$

□

The decomposition (56) is of course not unique. If we insist however that the number of columns of L be minimal, then L and R are unique up to right and left multiplication by constant invertible matrices respectively. Such decompositions are not of particular interest here. The two decompositions of particular interest in our case are:

$$H(s,\sigma) = L(s)\frac{\Sigma(\sigma)}{q(\sigma)} \tag{69}$$

$$H(s,\sigma) = \frac{S(s)}{p(s)}R(\sigma) \tag{70}$$

where $S(s)$ and $\Sigma(\sigma)$ are as in (61) and (62).

We say that the decomposition (69) is minimal if $q(\sigma)$ is a monic polynomial of least degree. If we require Σ be such that β_i's are as small as possible, then the minimal decomposition (69) becomes unique. We shall only consider this latter minimal decomposition which can be constructed as follows. First express H as

$$H(s,\sigma) = \frac{N(s,\sigma)}{p(s)q(\sigma)} \tag{71}$$

where the polynomial matrix $N(s,\sigma)$ and $q(\sigma)$ are coprime, i.e. there is no σ_0 such that $N(s,\sigma_0) = 0$ and $q(\sigma_0) = 0$. Then decompose $N(s,\sigma)$ as in (63) and let $L(s) = F(s)/p(s)$.

Note that $q(\sigma)$ is the monic polynomial which is the least common multiple of the denominators of all the entries of H in σ. And so if

$$\hat{L}(s)\frac{\hat{\Sigma}(\sigma)}{\hat{q}(\sigma)}$$

is any other decomposition of H, then $q(\sigma)$ divides $\hat{q}(\sigma)$.

Similarly, we can define and construct the minimal decomposition (70). The construction is as follows. Express H as in (71) with $N(s,\sigma)$ and $p(s)$ coprime in s. Then, decompose $N(s,\sigma)$ as in (64) and let $R(\sigma) = G(\sigma)/q(\sigma)$.

Definition 4.1 *$(C, A, \Gamma(\sigma))$ is an s-state-space representation of a strictly proper rational matrix having separable denominator $H(s,\sigma)$ if*

$$H(s,\sigma) = C(sI - A)^{-1}\Gamma(\sigma), \tag{72}$$

where C and A are constant matrices and $\Gamma(\sigma)$ a rational matrix in σ; we say that (72) is an s-factorization *of H.*

Similarly, $(\Omega(s), E, B)$ is a σ-state-space representation of a strictly proper rational matrix having separable denominator $H(s,\sigma)$ if

$$H(s,\sigma) = \Omega(s)(\sigma I - E)^{-1}B \tag{73}$$

for some constant matrices E and B, and a rational matrix $\Omega(s)$; we say that (73) is a σ-factorization *of H.*

Thanks to Theorem 4.1 any strictly proper rational matrix having separable denominator has both an s- and a σ-state-space representation. For example an s-state-space representation of a strictly proper rational matrix having separable denominator $H(s,\sigma)$ exists because thanks to Theorem 4.1, we can decompose H as in (56) and factor L (which is strictly proper) as follows

$$L(s) = C(sI - A)^{-1}B \tag{74}$$

which mean that $(C, A, BR(\sigma))$ is an s-state-space representation of H.

Definition 4.2 *The pair $(A, \Gamma(\sigma))$ is called reachable if for all constant left eigenvectors of A, v, $v\Gamma(\sigma) \neq 0$.*

The pair $(\Omega(s), E)$ is called observable if for all constant right eigenvectors of A, w, $\Omega(s)w \neq 0$.

Definition 4.2 generalizes the classical notions of reachability and observability in the 1D case.

Theorem 4.2 *$(C, A, \Gamma(\sigma))$ is a minimal s-state-space representation of $H(s,\sigma)$ if and only if*

1- (C, A) is observable,

2- $(A, \Gamma(\sigma))$ is reachable.

$(\Omega(s), E, B)$ is a minimal σ-state-space representation of $H(s,\sigma)$ if and only if

1- (E, B) is reachable,

2- $(\Omega(s), E)$ is observable.

Proof Suppose that $(C, A, \Gamma(\sigma))$ is minimal but $(A, \Gamma(\sigma))$ is not reachable and consider the minimal decomposition

$$\Gamma(\sigma) = K\frac{\Sigma(\sigma)}{q(\sigma)}, \tag{75}$$

(here K is a constant matrix because Γ is not a function of s). Then (A, K) is not reachable because for some left eigenvector of A, v, we have that $vK\frac{\Sigma(\sigma)}{q(\sigma)} = 0$ which implies that $vK = 0$. This means that there exist matrices $\hat{C}$, $\hat{A}$ and $\hat{K}$ with $\hat{A}$ having smaller dimension than A such that

$$C(sI - A)^{-1}K = \hat{C}(sI - \hat{A})^{-1}\hat{K}. \tag{76}$$

But

$$H(s,\sigma) = C(sI - A)^{-1}K\frac{\Sigma(\sigma)}{q(\sigma)} \tag{77}$$

which means that

$$H(s,\sigma) = \hat{C}(sI - \hat{A})^{-1}\hat{K}\frac{\Sigma(\sigma)}{q(\sigma)} \tag{78}$$

and so $(\hat{C}, \hat{A}, \hat{K}\frac{\Sigma(\sigma)}{q(\sigma)})$ is also an s-state-space representation of H which means that $(C, A, \Gamma(\sigma))$ is not minimal. Similarly, if (C, A) is not observable, we can find $\hat{C}$, $\hat{A}$ and $\hat{K}$ with $\hat{A}$ having smaller dimension than A such that (76) holds and again we get that $(C, A, \Gamma(\sigma))$ is not minimal.

Now suppose that $(C_i, A_i, \Gamma_i(\sigma))$, $i = 1, 2$, are two reachable and observable s-state-space representations, then we must show that they have equal dimensions. Consider the minimal decomposition

$$\left[\begin{array}{cc} \Gamma_1(\sigma) & \Gamma_2(\sigma) \end{array}\right] = \left[\begin{array}{cc} K_1 & K_2 \end{array}\right]\frac{\Sigma(\sigma)}{q(\sigma)} \tag{79}$$

then

$$H(s,\sigma) = C_1(sI - A_1)^{-1}K_1\frac{\Sigma(\sigma)}{q(\sigma)} = C_2(sI - A_2)^{-1}K_2\frac{\Sigma(\sigma)}{q(\sigma)}. \tag{80}$$

This implies that

$$C_1(sI - A_1)^{-1}K_1 = C_2(sI - A_2)^{-1}K_2, \tag{81}$$

but since (C_i, A_i) and (A_i, K_i), $i = 1, 2$, are observable and reachable, A_i's must have equal dimensions.

Minimality conditions for σ-state-space representations can be obtained in a similar fashion. □

A useful result which follows easily is the following:

Corollary *Let $(C_1, A_1, \Gamma_1(\sigma))$ and $(C_2, A_2, \Gamma_2(\sigma))$ be two minimal s-state-space representations of $H(s,\sigma)$. Then there exists an invertible matrix T such that*

$$\begin{aligned} \Gamma_2(\sigma) &= T\Gamma_1(\sigma) && (82) \\ C_2 &= C_1T^{-1} && (83) \\ A_2 &= TA_1T^{-1}. && (84) \end{aligned}$$

Let $(\Omega_1(s), E_1, B_1)$ and $(\Omega_2(s), E_2, B_2)$ be two minimal σ-state-space representations of $H(s,\sigma)$. Then there exists an invertible matrix U such that

$$\begin{aligned} B_2 &= UB_1 && (85) \\ \Omega_2(s) &= \Omega_1(s)U^{-1} && (86) \\ E_2 &= UE_1U^{-1}. && (87) \end{aligned}$$

For a rational matrix in one variable, the dimension of its minimal factorizations equals its *McMillan degree*, i.e. the degree of the least common multiple of the denominators of all of its minors. Similar results can be obtained for rational matrices in two variables having separable denominators.

Definition 4.3 *The s-degree of a strictly proper rational matrix with separable denominator $H(s,\sigma)$, denoted by $\mu_s(H(s,\sigma))$, is the minimal dimension of its s-state-space representations, i.e. if $(C, A, \Gamma(\sigma))$ is a minimal s-state-space representation of $H(s,\sigma)$, then $\mu_s(H(s,\sigma)) = \dim A$.*

The σ-degree of $H(s,\sigma)$, denoted by $\mu_\sigma(H(s,\sigma))$, is the minimal of its σ-state-space representations, i.e. if $(\Omega(s), E, B)$ is a minimal σ-state-space representation of $H(s,\sigma)$, then $\mu_\sigma(H(s,\sigma)) = \dim E$.

Note that in general, the s-degree is not equal to the McMillan degree in s and that the σ-degree is not equal to the McMillan degree in σ. The McMillan degree in s for example corresponds to the dimension of minimal factorizations of the following type:

$$H(s,\sigma) = C(\sigma)(sI - A(\sigma))^{-1}B(\sigma) \tag{88}$$

which is a more general factorization than (72) where we require that C and A be constant. The additional degrees of freedom in (88) imply that the s-degree is always larger than or equal to the McMillan degree in s. Similarly, the σ-degree is always larger than or equal to the McMillan degree in σ.

Example 4.1 Consider

$$H(s,\sigma) = \begin{bmatrix} 1/\sigma^2 s \\ 1/\sigma s \end{bmatrix}. \tag{89}$$

It is straightforward to verify that the s-state-representation of $H(s,\sigma)$,

$$(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/\sigma^2 \\ 1/\sigma \end{bmatrix})$$

is minimal and thus $\mu_s(H(s,\sigma)) = 2$, whereas, the McMillan degree in s of $H(s,\sigma)$ is 1 since we can express H as follows:

$$H(s,\sigma) = \begin{bmatrix} 1/\sigma^2 \\ 1/\sigma \end{bmatrix} (s)^{-1} 1. \tag{90}$$

Theorem 4.3 *Consider the minimal decomposition*

$$H(s,\sigma) = L(s)\frac{\Sigma(\sigma)}{q(\sigma)}. \tag{91}$$

Then the s-degree of H, $\mu_s(H(s,\sigma))$, equals the McMillan degree of $L(s)$.

Consider the minimal decomposition

$$H(s,\sigma) = \frac{S(s)}{p(s)}R(\sigma). \tag{92}$$

Then the σ-degree of H, $\mu_\sigma(H(s,\sigma))$, equals the McMillan degree of $R(\sigma)$.

Proof Clearly $\mu_s(H(s,\sigma))$ is less than or equal to the McMillan degree of $L(s)$ because we can factor $L(s)$ as follows

$$L(s) = C(sI - A)^{-1}B \tag{93}$$

with dimension of A being the McMillan degree of $L(s)$ and since

$$H(s,\sigma) = C(sI - A)^{-1}B\frac{\Sigma(\sigma)}{q(\sigma)} \tag{94}$$

we get that $(C, A, B\frac{\Sigma(\sigma)}{q(\sigma)})$ is an s-state-space representation of H.

To show that $\mu_s(H(s,\sigma))$ is larger than or equal to the McMillan degree of $L(s)$, first we need to show that if $(C, A, \Gamma(\sigma))$ is a minimal s-state-space representation of H, and

$$\Gamma(\sigma) = K\frac{\hat{\Sigma}(\sigma)}{\hat{q}(\sigma)} \tag{95}$$

a minimal decomposition, then

$$\hat{\Sigma}(\sigma) = \Sigma(\sigma) \tag{96}$$
$$\hat{q}(\sigma) = q(\sigma). \tag{97}$$

To show this note that

$$H(s,\sigma) = C(sI - A)^{-1}\Gamma(\sigma) = L(s)\frac{\Sigma(\sigma)}{q(\sigma)}. \tag{98}$$

If we now do a power series expansion in s and consider the first n-terms, we get that

$$O\Gamma(\sigma) = \Lambda\frac{\Sigma(\sigma)}{q(\sigma)} \tag{99}$$

where O denotes the observability matrix of (C, A) and Λ is a constant matrix composed of the first n coefficients in the expansion of $L(s)$. Since (C, A) is observable, O has full column rank and thus a left inverse which we denote O^{-l}. If we now premultiply both sides of (99) with O^{-l}, we get

$$\Gamma(\sigma) = O^{-l}\Lambda\frac{\Sigma(\sigma)}{q(\sigma)}. \tag{100}$$

Clearly (100) is a decomposition of $\Gamma(\sigma)$ and since (95) is the minimal decomposition, we can deduce that $\hat{q}(\sigma)$ divides $q(\sigma)$. On the other hand,

$$H(s,\sigma) = C(sI - A)^{-1}K\frac{\hat{\Sigma}(\sigma)}{\hat{q}(\sigma)} \tag{101}$$

is a decomposition of $H(s,\sigma)$ which since (91) is a minimal decomposition of H implies that $q(\sigma)$ divides $\hat{q}(\sigma)$. Which, since $q(\sigma)$ and $\hat{q}(\sigma)$ are monic polynomials, implies that they are equal. To show that $\hat{\Sigma}$ and Σ are equal, simply note that since $q(\sigma)$ and $\hat{q}(\sigma)$ are equal, from (98) we get that β_i and $\hat{\beta}_i$ correspond to the highest power of σ in the ith column of $q(\sigma)H(s,\sigma)$. Thus $\beta_i = \hat{\beta}_i$ for all i which clearly implies (96).

Continuing the proof, note that from (95-98) we get

$$L(s) = C(sI - A)^{-1}K \tag{102}$$

which means that the dimension of A, which is equal to $\mu_s(H(s,\sigma))$, is larger than or equal to the McMillan degree of $L(s)$. Thus, the McMillan degree of $L(s)$ equals $\mu_s(H(s,\sigma))$.

The second part of the theorem can be proved similarly. □

An important consequence of Theorem 4.3 is that to construct minimal s- and σ-factorizations, we simply have to decompose H as in (91) and (92), and then factor L and R using standard algorithms.

5 Realizability conditions

In Section 2, we defined the transforms of the causal and anti-causal parts of the weighting pattern of an autonomous, boundary-value linear system. We noted the transform of the causal part by $W_c(s,\sigma)$ and the anti-causal part by $W_a(s,\sigma)$. We have already seen that, to be realizable, W_c and W_a must be rational and strictly proper. However, these conditions are not sufficient for realizability. The following theorem characterizes completely the class of realizable matrix pairs.

Theorem 5.1 *The pair of rational matrices* $\{W_c(s,\sigma), W_a(s,\sigma)\}$ *is realizable if and only if*

1- the two matrices are strictly proper and have separable denominators,

2- there exists a strictly proper rational matrix $H(z)$ *such that*

$$(s+\sigma)(W_c(s,\sigma)-W_a(s,\sigma)) = H(s)-H(-\sigma). \tag{103}$$

Proof The necessity of condition 1 follows easily expressions (10) and (11). The necessity of condition 2 is also immediate, simply note that we can choose H as

$$H(z) = C(zI-A)^{-1}B. \tag{104}$$

To show sufficiency, we suppose that W_c and W_a satisfy conditions 1 and 2, and we find (C,P,A,B) such that (10) and (11) are satisfied.

We can express $W_a(s,\sigma)$ as (see Theorem 4.1)

$$W_a(s,\sigma) = L_a(s)R_a(\sigma) \tag{105}$$

where L_a and R_a are strictly proper rational matrices and thus can be factored as

$$\begin{aligned} L_a(s) &= C_1(sI-A_1)^{-1}B_1 & (106)\\ R_a(\sigma) &= C_2(\sigma I-A_2)^{-1}B_2, & (107)\end{aligned}$$

which implies that W_a can be expressed as follows

$$W_a(s,\sigma) = C_1(sI-A_1)^{-1}B_1C_2(\sigma I-A_2)^{-1}B_2. \tag{108}$$

Matrix H is supposed to satisfy (103) and be strictly proper. Since it is strictly proper, it can be factored as follows

$$H(z) = C(zI-A)^{-1}B \tag{109}$$

which implies that

$$\begin{aligned} W_c(s,\sigma)-W_a(s,\sigma) &= \frac{1}{s+\sigma}(H(s)-H(-\sigma)) \\ &= C(sI-A)^{-1}(\sigma I+A)^{-1}B. & (110)\end{aligned}$$

From (108) and (110), it is a simple exercise to show that (C, P, A, B) where

$$C = \begin{pmatrix} C_1 & 0 & C \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & B_1C_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & -A_2 & 0 \\ 0 & 0 & A \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ B_2 \\ B \end{pmatrix}, \tag{111}$$

realizes the pair $\{W_c, W_a\}$. □

Condition 2 simply reflects the fact that $w_c(t,\tau) - w_a(t,\tau)$ depends only on the difference $t - \tau$. This will become clearer in Section 7 when we study the class of stationary systems.

Theorem 5.1 provides us with a simple realizability test. Given a candidate weighting pattern $w(t,\tau)$, compute its transforms as in (10) and (11), and see if they satisfy the two conditions of Theorem 5.1. Since w is made of two analytic functions, we can consider realizing w on different intervals. The following result shows that the choice of the interval over which w is to be realized does not matter as far as realizability is concerned.

Theorem 5.2 *If w is realizable on any interval, it is realizable on all intervals.*

Proof Suppose that w is realizable on $[a,b]$, which means that it has a realization (C, P, A, B), and let $[c,d]$ be any interval. Then what we have to show is that w is also realizable on $[c,d]$, i.e., we have to show that the transform of w on this new interval, $\{\hat{W}_c, \hat{W}_a\}$, is a realizable pair. But

$$\begin{aligned}
\hat{W}_c(s,\sigma) &= \int_c^\infty \int_c^\infty w_c(t,\tau) e^{-s(t-c)-\sigma(\tau-c)} dt d\tau \\
&= \int_c^\infty \int_c^\infty C \exp((t-a)A)(I-P)\exp(-(\tau-a)A) B e^{-s(t-c)-\sigma(\tau-c)} \\
&= C(sI-A)^{-1}(I-\hat{P})(\sigma I + A)^{-1} B, \qquad\qquad (112) \\
\hat{W}_a(s,\sigma) &= \int_c^\infty \int_c^\infty w_a(t,\tau) e^{-s(t-a)-\sigma(\tau-a)} dt d\tau \\
&= \int_c^\infty \int_c^\infty -C \exp((t-a)A) P \exp(-(\tau-a)A) B e^{-s(t-a)-\sigma(\tau-a)} dt d\tau \\
&= -C(sI-A)^{-1}\hat{P}(\sigma I + A)^{-1} B \qquad\qquad (113)
\end{aligned}$$

where $\hat{P} = \exp((a-c)A) P \exp((c-a)A$, which clearly indicates that $(C, \hat{P}, A, B)$ is a realization of $\{\hat{W}_c, \hat{W}_a\}$. □

Example 5.1 Let us examine the realizability of

$$w(t,\tau) = \begin{cases} \cos(t)\cos(\tau), & 0 \le \tau < t \le \pi, \\ -\sin(t)\sin(\tau), & 0 \le t < \tau \le \pi, \end{cases} . \tag{114}$$

In this case,

$$W_c(s,\sigma) = \frac{s\sigma}{(s^2+1)(\sigma^2+1)} \tag{115}$$

$$W_a(s,\sigma) = -\frac{1}{(s^2+1)(\sigma^2+1)}. \tag{116}$$

The first realizability condition is clearly satisfied. The second condition is also verified because

$$\begin{aligned}(s+\sigma)(W_c(s,\sigma)-W_a(s,\sigma)) &= (s+\sigma)\frac{(s\sigma+1)}{(s^2+1)(\sigma^2+1)} \\ &= \frac{s}{s^2+1} - \frac{-\sigma}{(-\sigma)^2+1}.\end{aligned} \tag{117}$$

Thus (114) is realizable.

In the proof of Theorem 5.1, we have obtained a method for constructing a realization of any realizable pair $\{W_c, W_a\}$. The problem with this procedure is that, in general, it yields a non minimal realization. Of course minimal realizations can be constructed from non minimal realizations [5], but the procedure is complicated. Our goal in this paper is to construct minimal realizations directly by performing standard rational matrix factorizations.

Example 5.2 Consider the problem of realizing the pair $\{1/s\sigma, -1/s\sigma\}$. Clearly $1/s\sigma$ and $-1/s\sigma$ are strictly proper in s and σ, and so they meet the first realizability condition. They also satisfy the second condition because

$$(s+\sigma)(1/s\sigma + 1/s\sigma) = 2/s + 2/\sigma = 2/s - 2/(-\sigma). \tag{118}$$

Thus this pair is realizable. If we follow the procedure used in the proof of Theorem 5.1 for constructing a realization, we get the following realization

$$(\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix})$$

But this realization is not minimal because $(2, 1/2, 0, 1)$ is also a realization of $\{1/s\sigma, -1/s\sigma\}$.

6 Realization procedure

In this section, we derive an expression for the dimension of minimal realizations. We then use this result to solve the realization problem via methods of factorization of rational matrices introduced in Section 4.

Theorem 6.1 *The dimension n of minimal realizations of a realizable pair $\{W_c(s,\sigma), W_a(s,\sigma)\}$ is given by the following equation*

$$n = \mu_s(\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix}) + \mu_\sigma(\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix}) - \mu_s(W_c(s,\sigma) - W_a(s,\sigma)) \tag{119}$$

where $\mu_s(X)$ and $\mu_\sigma(X)$ denote respectively the s- and σ-degrees of X.

Proof Suppose (C, P, A, B) is a minimal realization of $\{W_c(s,\sigma), W_a(s,\sigma)\}$ then

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix} = C(sI-A)^{-1} \begin{bmatrix} (I-P)(\sigma I+A)^{-1}B & -P(\sigma I+A)^{-1}B \end{bmatrix}. \tag{120}$$

We start by showing that $\mu_s(\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix})$ equals the rank of the observability matrix of (C, A). Note that thanks to the second minimality condition (Theorem 3.1),

$$(A, \begin{bmatrix} (I-P)(\sigma I+A)^{-1}B & -P(\sigma I+A)^{-1}B \end{bmatrix})$$

is reachable and so if

$$\begin{bmatrix} (I-P)(\sigma I+A)^{-1}B & -P(\sigma I+A)^{-1}B \end{bmatrix} = K\frac{\Sigma(\sigma)}{q(\sigma)} \tag{121}$$

is a minimal decomposition, then (A, K) is reachable. This implies that there exist matrices $\hat{C}$, $\hat{A}$ and $\hat{B}$ such that

$$C(sI-A)^{-1}K = \hat{C}(sI-\hat{A})^{-1}\hat{B} \tag{122}$$

with the dimension of $\hat{A}$ being equal to the rank of the observability matrix (C, A) and where $(\hat{C}, \hat{A})$ and $(\hat{A}, \hat{B})$ are respectively observable and reachable. But from (121) and (122) follows that

$$(\hat{C}, \hat{A}, \hat{B}\frac{\Sigma(\sigma)}{q(\sigma)})$$

is an s-state-space representation of $\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix}$ which means that the s-degree of $\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix}$ is less than or equal to the rank of the observability matrix of (C, A). On the other hand, it is straightforward to verify that since $(\hat{A}, \hat{B})$ is reachable, $(\hat{A}, \hat{B}\frac{\Sigma(\sigma)}{q(\sigma)})$ is reachable and thus, thanks to Theorem 4.2, $(\hat{C}, \hat{A}, \hat{B}\frac{\Sigma(\sigma)}{q(\sigma)})$ is a minimal s-state-space representation. Thus,

$$\mu_s(\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix}) = rank(O), \tag{123}$$

where O denotes the observability matrix of (C, A).

Similarly, we can show that

$$\mu_\sigma(\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix}) = rank(R), \tag{124}$$

where R denotes the reachability matrix of (A, B).

Now we show that

$$\mu_s(W_c(s,\sigma) - W_a(s,\sigma)) = rank(OR). \tag{125}$$

Let

$$W(s,\sigma) = W_c(s,\sigma) - W_a(s,\sigma) = C(sI - A)^{-1}(\sigma I + A)^{-1}B \tag{126}$$

and consider the minimal decomposition

$$(\sigma I + A)^{-1}B = K\frac{\Sigma(\sigma)}{q(\sigma)}. \tag{127}$$

Then the reachability space of (A, B) equals Im K because for any vector v,

$$v(\sigma I + A)^{-1}B = 0 \tag{128}$$

if and only if

$$vK = 0. \tag{129}$$

This implies that the reachability space of (A, K) equals the reachability space of (A, B) and thus there exist matrices $\hat{C}$, $\hat{A}$ and $\hat{B}$ such that

$$C(sI - A)^{-1}K = \hat{C}(sI - \hat{A})^{-1}\hat{B} \tag{130}$$

with the dimension of $\hat{A}$ being equal to the rank of OR and $(\hat{C}, \hat{A})$ and $(\hat{A}, \hat{B})$ being respectively observable and reachable. Thus

$$(\hat{C}, \hat{A}, \hat{B}\frac{\Sigma(\sigma)}{q(\sigma)})$$

is an s-state-space representation of $W(s,\sigma)$ and so the s-degree of $W(s,\sigma)$ is less than or equal to the rank of OR. On the other hand, it is straightforward to verify that $(\hat{A}, \hat{B}\frac{\Sigma(\sigma)}{q(\sigma)})$ is reachable and thus, thanks to Theorem 4.2, we can show (125).

Finally, because of the third minimality condition (14),

$$rank(OR) = n - rank(O) - rank(R), \tag{131}$$

which thanks to (123) and (124) implies (119). $\square$

Note that

$$\mu_s(W_c(s,\sigma) - W_a(s,\sigma)) = \mu_\sigma(W_c(s,\sigma) - W_a(s,\sigma)). \tag{132}$$

This follows easily the second condition of realizability in Theorem 5.1.

In addition to the dimension of minimal realizations, (119) allows us to test the reachability and observability of minimal realizations. In particular, if

$$\mu_s(\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix})$$

which equals the rank of the observability matrix is equal to n, we can conclude that minimal realizations are observable. On the other hand, if

$$\mu_\sigma(\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix})$$

equals n, then minimal realizations are reachable.

Example 6.1 Consider the weighting pattern

$$w(t,\tau) = t\tau, \quad 0 \leq t,\tau \leq 1. \tag{133}$$

The transform of this weighting pattern is the pair $\{1/s^2\sigma^2, 1/s^2\sigma^2\}$ which is clearly realizable. Applying Theorem 6.1, we see that minimal realizations of (133) are not reachable or observable; the dimension of minimal realizations is 2 whereas the rank of their reachability and observability matrices are 1.

If minimal realizations are reachable or observable, the realization problem is simple; it only involves one factorization. The situation is more complicated when minimal realizations are not reachable or observable. Let us start with the case where minimal realizations are observable.

Observable case Suppose that minimal realizations are observable, i.e.

$$\mu_s(\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix}) = n. \tag{134}$$

Then, a minimal realization can be constructed as follows:

1- perform a minimal s-factorization

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix} = C(sI-A)^{-1} \begin{bmatrix} \Lambda_1(\sigma) & \Lambda_2(\sigma) \end{bmatrix} \tag{135}$$

2- compute

$$B = \lim_{\sigma\to\infty} \sigma(\Lambda_1(\sigma) - \Lambda_2(\sigma)). \tag{136}$$

3- construct the minimal decomposition

$$\begin{bmatrix} \Lambda_2(\sigma) \\ (\sigma I + A)^{-1}B \end{bmatrix} = \begin{bmatrix} Y \\ M \end{bmatrix} \frac{\Sigma(\sigma)}{q(\sigma)} \tag{137}$$

and find a P that satisfies

$$PM = -Y. \tag{138}$$

Claim (C, P, A, B) *is a minimal realization of* $\{W_c, W_a\}$.

Proof Let (C_r, P_r, A_r, B_r) be a minimal realization of $\{W_c, W_a\}$. What we have to show is that (C, P, A, B) and (C_r, P_r, A_r, B_r) are related as in Theorem 3.2.

Note that since (C_r, A_r) is observable,

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix} = C_r(sI - A_r)^{-1} \begin{bmatrix} (I - P_r)(\sigma I + A_r)^{-1}B_r & -P_r(\sigma I + A_r)^{-1}B_r \end{bmatrix} \tag{139}$$

is a minimal s-factorization and thus thanks to the corollary of Theorem 4.2, there exists a matrix T such that

$$C = C_r T^{-1} \tag{140}$$

$$A = T A_r T^{-1} \tag{141}$$

$$\begin{bmatrix} \Lambda_1(\sigma) & \Lambda_2(\sigma) \end{bmatrix} = T \begin{bmatrix} (I - P_r)(\sigma I + A_r)^{-1}B_r & -P_r(\sigma I + A_r)^{-1}B_r \end{bmatrix}. \tag{142}$$

Also from (136) and (142) follows that

$$B = T B_r. \tag{143}$$

Finally, notice that

$$\Lambda_2(\sigma) = -T P_r T^{-1}(\sigma I + A)^{-1} B \tag{144}$$

which implies that

$$\begin{aligned} \begin{bmatrix} \Lambda_2(\sigma) \\ (\sigma I + A)^{-1}B \end{bmatrix} &= \begin{bmatrix} Y \\ M \end{bmatrix} \frac{\Sigma(\sigma)}{q(\sigma)} \\ &= \begin{bmatrix} -T P_r T^{-1} \\ I \end{bmatrix} (\sigma I + A)^{-1} B, \end{aligned} \tag{145}$$

from which we get that

$$T P_r T^{-1} M = -Y \tag{146}$$

(this shows that (138) has at least one solution). Now, from (138) and (146) follows that

$$(P - T P_r T^{-1}) M = 0 \tag{147}$$

which implies that

$$(P - T P_r T^{-1}) M \frac{\Sigma(\sigma)}{q(\sigma)} = (P - T P_r T^{-1})(\sigma I + A)^{-1} B = 0, \tag{148}$$

which in turn implies that

$$(P - T P_r T^{-1}) R = 0. \tag{149}$$

By combining (140), (141), (143) and (149), and using the corollary of Theorem 3.2, we see that (C, P, A, B) is a minimal realization. □

Reachable case Suppose that minimal systems are reachable. i.e.

$$\mu_\sigma(\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix}) = n, \tag{150}$$

then a minimal factorization can be constructed as follows

1- perform a minimal σ-factorization

$$\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix} = \begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \end{bmatrix} (\sigma I + A)^{-1} B. \tag{151}$$

2- compute

$$C = \lim_{s \to \infty} s(\Psi_1(s) - \Psi_2(s)). \tag{152}$$

3- construct the minimal decomposition

$$\begin{bmatrix} C(sI - A)^{-1} & \Psi_2(s) \end{bmatrix} = \frac{S(s)}{p(s)} \begin{bmatrix} Y & M \end{bmatrix} \tag{153}$$

and find a P satisfying

$$MP = -Y. \tag{154}$$

Claim *(C, P, A, B) is a minimal realization of $\{W_c, W_a\}$.*

The proof is similar to the observable case and is omitted.

Unreachable/unobservable case If minimal realizations are not reachable or observable, the situation is more complicated because the matrix A cannot be constructed by just one factorization as in the previous cases. In this case, we have to construct it from two different pieces which are obtained from two different factorizations.

The following procedure can be used for constructing a minimal realization:

1- construct minimal s- and σ-factorizations

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix} = \bar{C}(sI - \bar{A})^{-1} \begin{bmatrix} \Lambda_c(\sigma) & \Lambda_a(\sigma) \end{bmatrix} \tag{155}$$

$$\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix} = \begin{bmatrix} \Psi_c(s) \\ \Psi_a(s) \end{bmatrix} (\sigma I + \tilde{A})^{-1} \tilde{B} \tag{156}$$

and let

$$\Psi = \lim_{s \to \infty} s(\Psi_c(s) - \Psi_a(s)) \tag{157}$$

$$\Lambda = \lim_{\sigma \to \infty} \sigma(\Lambda_c(\sigma) - \Lambda_a(\sigma)), \tag{158}$$

2- decompose $(\bar{C}, \bar{A}, \Lambda)$ in reachable/unreachable parts and $(\Psi, \tilde{A}, \tilde{B})$ in observable/unobservable parts:

$$\bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, \tag{159}$$

and

$$\Psi = \begin{bmatrix} 0 & \tilde{C}_2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \tag{160}$$

3- construct system matrices as follows

$$A = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 V^{-1} & \star \\ 0 & \bar{A}_1 & \bar{A}_2 \\ 0 & 0 & \bar{A}_4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \bar{C}_1 & \bar{C}_2 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{B}_1 \\ \bar{B}_1 \\ 0 \end{bmatrix}, \tag{161}$$

where $\star$ indicates an arbitrary block entry and the matrix V is given by

$$V = \bar{R}\tilde{R}(\tilde{R}\tilde{R}^T)^{-1}, \tag{162}$$

where $\bar{R}$ and $\tilde{R}$ denote respectively the reachability matrices of $(\bar{A}_1, \bar{B}_1)$ and $(\tilde{A}_4, \tilde{B}_2)$.

4- construct the minimal decompositions

$$\begin{bmatrix} W_a(s,\sigma) & C(sI-A)^{-1} \end{bmatrix} = \frac{S(s)}{p(s)} \begin{bmatrix} G(\sigma) & M_1 \end{bmatrix} \tag{163}$$

$$\begin{bmatrix} G(\sigma) & (\sigma I + A)^{-1} B \end{bmatrix} = \begin{bmatrix} Y & M_2 \end{bmatrix} \frac{\Sigma(\sigma)}{q(\sigma)} \tag{164}$$

and find a P that satisfies

$$M_1 P M_2 = -Y. \tag{165}$$

Claim *(C, P, A, B) is a minimal realization of $\{W_c, W_a\}$.*

Proof Consider a minimal realization (C_r, P_r, A_r, B_r) where (C_r, A_r, B_r) is in the 4-part Kalman decomposed form. In this case, because of the third minimality condition (Theorem 3.1), there is no unreachable *and* unobservable part. So, we have

$$C_r = \begin{bmatrix} 0 & C_{r,2} & C_{r,3} \end{bmatrix}, \tag{166}$$

$$B_r = \begin{bmatrix} B_{r,1} \\ B_{r,2} \\ 0 \end{bmatrix}, \tag{167}$$

$$A_r = \begin{bmatrix} A_{r,1} & A_{r,4} & A_{r,6} \\ 0 & A_{r,2} & A_{r,5} \\ 0 & 0 & A_{r,3} \end{bmatrix}, \tag{168}$$

$$P_r = \begin{bmatrix} P_{r,11} & P_{r,12} & P_{r,13} \\ P_{r,21} & P_{r,22} & P_{r,23} \\ P_{r,31} & P_{r,32} & P_{r,33} \end{bmatrix}. \tag{169}$$

By direct calculation, we can show that

$$\left(\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma) \end{bmatrix}\right) = \begin{bmatrix} J_c(s) \\ J_a(s) \end{bmatrix}(\sigma I + \begin{bmatrix} A_{r,1} & A_{r,4} \\ 0 & A_{r,2} \end{bmatrix})^{-1} \begin{bmatrix} B_{r,1} \\ B_{r,2} \end{bmatrix}, \tag{170}$$

where

$$J_c(s) - J_a(s) = \begin{bmatrix} 0 & C_{r,2} \end{bmatrix}(sI - \begin{bmatrix} A_{r,1} & A_{r,4} \\ 0 & A_{r,2} \end{bmatrix})^{-1}. \tag{171}$$

Similarly

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix} = \begin{bmatrix} C_{r,2} & C_{r,3} \end{bmatrix}(sI - \begin{bmatrix} A_{r,2} & A_{r,5} \\ 0 & A_{r,3} \end{bmatrix})^{-1} \begin{bmatrix} K_c(\sigma) & K_a(\sigma) \end{bmatrix}, \tag{172}$$

where

$$K_c(\sigma) - K_a(\sigma) = (\sigma I + \begin{bmatrix} A_{r,2} & A_{r,5} \\ 0 & A_{r,3} \end{bmatrix})^{-1} \begin{bmatrix} B_{r,2} \\ 0 \end{bmatrix}. \tag{173}$$

Since (170) and (172) are minimal σ- and s-factorizations, thanks to the corollary of Theorem 4.2, $(\bar{C}, \bar{A}, \begin{bmatrix} \Lambda_c(\sigma) & \Lambda_a(\sigma) \end{bmatrix})$ and $(\begin{bmatrix} \Psi_c(s) & \Psi_a(s) \end{bmatrix}, \tilde{A}, \tilde{B})$ are related respectively to

$$(\begin{bmatrix} C_{r,2} & C_{r,3} \end{bmatrix}, \begin{bmatrix} A_{r,2} & A_{r,5} \\ 0 & A_{r,3} \end{bmatrix}, \begin{bmatrix} K_c(\sigma) & K_a(\sigma) \end{bmatrix})$$

and

$$(\begin{bmatrix} J_c(s) \\ J_a(s) \end{bmatrix}, \begin{bmatrix} A_{r,1} & A_{r,4} \\ 0 & A_{r,2} \end{bmatrix}, \begin{bmatrix} B_{r,1} \\ B_{r,2} \end{bmatrix})$$

by similarity transformations. It follows then that $(\bar{C}, \bar{A}, \Lambda)$ and $(\Psi, \tilde{A}, \tilde{B})$ are related respectively to

$$(\begin{bmatrix} C_{r,2} & C_{r,3} \end{bmatrix}, \begin{bmatrix} A_{r,2} & A_{r,5} \\ 0 & A_{r,3} \end{bmatrix}, \begin{bmatrix} B_{r,2} \\ 0 \end{bmatrix})$$

and

$$(\begin{bmatrix} 0 & C_{r,2} \end{bmatrix}, \begin{bmatrix} A_{r,1} & A_{r,4} \\ 0 & A_{r,2} \end{bmatrix}, \begin{bmatrix} B_{r,1} \\ B_{r,2} \end{bmatrix})$$

by the same similarity transformations, W and U. Because of the structure of the two factorizations, it is easy to show that W and U must be upper block tri-diagonal:

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_4 \end{bmatrix} \tag{174}$$

$$U = \begin{bmatrix} U_1 & U_2 \\ 0 & U_4 \end{bmatrix}. \tag{175}$$

Now notice that

$$\begin{aligned} W_c(s,\sigma) - W_a(s,\sigma) &= \bar{C}(sI - \bar{A})^{-1}[\Lambda_c(\sigma) - \Lambda_a(\sigma)] \\ &= [\Psi_c(s) - \Psi_a(s)](\sigma I + \tilde{A})^{-1}\tilde{B}. \end{aligned} \tag{176}$$

From the second realizability condition (Theorem 5.1), follows that

$$W_c(s,\sigma) - W_a(s,\sigma) = W_c(-\sigma,-s) - W_a(-\sigma,-s) \tag{177}$$

and so

$$\bar{C}(sI-\bar{A})^{-1}[\Lambda_c(\sigma)-\Lambda_a(\sigma)] = [\Psi_c(-\sigma)-\Psi_a(-\sigma)](-sI+\tilde{A})^{-1}\tilde{B} \tag{178}$$

which by multiplying both sides by σ and taking the limit as $\sigma \to \infty$ yields

$$\bar{C}(sI-\bar{A})^{-1}\Lambda = \Psi(s)(sI-\tilde{A})^{-1}\tilde{B}, \tag{179}$$

which implies that

$$W_c(s,\sigma) - W_a(s,\sigma) = \bar{C}_1(sI-\bar{A}_1)^{-1}\bar{B}_1 = \tilde{C}_2(sI-\tilde{A}_4)^{-1}\tilde{B}_2. \tag{180}$$

But factorizations $(\bar{C}_1,\bar{A}_1,\bar{B}_1)$ and $(\tilde{C}_2,\tilde{A}_4,\tilde{B}_2)$ are reachable and observable and thus

$$\bar{C}_1 = \tilde{C}_2V^{-1}, \quad \bar{A}_1 = V\tilde{A}_4V^{-1}, \quad \bar{B}_1 = V\tilde{B}_2 \tag{181}$$

where V is defined in (162).

Now it can be shown by direct calculation using (162) that (C,A,B) and (C_r,A_r,B_r) are related as in Theorem 3.2 with

$$T = \begin{bmatrix} U_1 & U_2V^{-1} & \star \\ 0 & W_1 & W_2 \\ 0 & 0 & W_4 \end{bmatrix}, \tag{182}$$

so all that remains to be shown is that P and P_r are also related as required. Note that since

$$\begin{aligned} W_a(s,\sigma) &= -C_r(sI-A_r)^{-1}P_r(\sigma I+A_r)^{-1}B_r \\ &= -C(sI-A)^{-1}T^{-1}P_rT(\sigma I+A)^{-1}B, \end{aligned} \tag{183}$$

(163) implies that

$$G(\sigma) = -M_1T^{-1}P_rT(\sigma I+A)^{-1}B. \tag{184}$$

From (164) and (184) follows that

$$M_1T^{-1}P_rTM_2 = -Y, \tag{185}$$

and so from (165) we get

$$M_1(P-T^{-1}P_rT)M_2 = 0. \tag{186}$$

This implies that

$$\frac{S(s)}{p(s)}M_1(P-T^{-1}P_rT)M_2\frac{\Sigma(\sigma)}{q(\sigma)} = C(sI-A)^{-1}(P-T^{-1}P_rT)(\sigma I+A)^{-1}B = 0, \tag{187}$$

which finally implies

$$O(P-T^{-1}P_rT)R = 0 \tag{188}$$

which means that (C,P,A,B) and (C_r,P_r,A_r,B_r) are related as in Theorem 3.2 and thus thanks to the corollary of Theorem 3.2, (C,P,A,B) is a minimal realization. $\square$

Example 6.2 Consider the problem of realizing

$$w(t,\tau)=\begin{cases}\begin{bmatrix} t\tau+t-\tau \\ 0\end{bmatrix}, & 0\le\tau<t\le\pi, \\ \begin{bmatrix} t\tau \\ t-\tau\end{bmatrix}, & 0\le t<\tau\le\pi,\end{cases}. \tag{189}$$

The pair $\{W_c,W_a\}$ is given by

$$W_c(s,\sigma) = \begin{bmatrix}(1-s+\sigma)/s^2\sigma^2 \\ 0\end{bmatrix} \tag{190}$$

$$W_a(s,\sigma) = \begin{bmatrix}1/s^2\sigma^2 \\ (-s+\sigma)/s^2\sigma^2\end{bmatrix}. \tag{191}$$

We start by examining the realizability of $\{W_c,W_a\}$. The first realizability condition (Theorem 5.1) is clearly satisfied. The second condition is also satisfied because

$$\begin{aligned}(s+\sigma)(W_c(s,\sigma)-W_a(s,\sigma)) &= (s+\sigma)\begin{bmatrix}(-s+\sigma)/s^2\sigma^2 \\ (s-\sigma)/s^2\sigma^2\end{bmatrix} \\ &= \begin{bmatrix}-1/s^2 \\ 1/s^2\end{bmatrix}-\begin{bmatrix}-1/(-\sigma)^2 \\ 1/(-\sigma)^2\end{bmatrix}.\end{aligned} \tag{192}$$

Next, we have to find the dimension of minimal realizations. For that, we need to compute (119) and thus we need to construct the following minimal decompositions

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma)\end{bmatrix} = \begin{bmatrix}1/s^2-1/s & 1/s^2 & 1/s^2 & 0 \\ 0 & 0 & -1/s & 1/s^2\end{bmatrix}\begin{bmatrix}1 & 0 \\ \sigma & 0 \\ 0 & 1 \\ 0 & \sigma\end{bmatrix}\frac{1}{\sigma^2}, \tag{193}$$

$$\begin{bmatrix} W_c(s,\sigma) \\ W_a(s,\sigma)\end{bmatrix} = \frac{1}{s^2}\begin{bmatrix}1 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & s\end{bmatrix}\begin{bmatrix}1/\sigma \\ -1/\sigma^2 \\ 1/\sigma^2 \\ 1/\sigma \\ -1/\sigma^2\end{bmatrix}, \tag{194}$$

$$W_c(s,\sigma)-W_a(s,\sigma) = \begin{bmatrix}-1/s & 1/s^2 \\ 1/s & -1/s^2\end{bmatrix}\begin{bmatrix}1 \\ \sigma\end{bmatrix}\frac{1}{\sigma^2}. \tag{195}$$

Using Theorem 4.3, we can show that (119) becomes

$$n=4+2-2=4, \tag{196}$$

from which we conclude that minimal realizations have dimension 4 and are observable. A minimal realization (C,P,A,B) can then be constructed as follows.

1- A minimal s-factorization (135) is constructed by factoring the s part of (193) using standard algorithms; we obtain

$$C = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \tag{197}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{198}$$

$$\begin{bmatrix} \Lambda_1(\sigma) & \Lambda_2(\sigma) \end{bmatrix} = \begin{bmatrix} -1/\sigma^2 & 0 \\ 1/\sigma^2 + 1/\sigma & 1/\sigma \\ 0 & 1/\sigma^2 - 1/\sigma \\ 0 & -1/\sigma \end{bmatrix}, \tag{199}$$

2- from (199) we get

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \tag{200}$$

3- finally we construct

$$\begin{bmatrix} 0 \\ 1/\sigma \\ 1/\sigma^2 - 1/\sigma \\ -1/\sigma \\ 0 \\ 0 \\ 1/\sigma - 1/\sigma^2 \\ 1/\sigma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sigma \end{bmatrix} \frac{1}{\sigma^2}, \tag{201}$$

which implies that P is any matrix satisfying

$$P \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{202}$$

which implies that

$$P = \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & -1 \\ \star & \star & 1 & 0 \\ \star & \star & 0 & 1 \end{bmatrix} \tag{203}$$

where $\star$'s are arbitrary elements.

If the weighting pattern which is to be realized is stationary, i.e. $w(t, \tau)$ is only a function of the difference $t - \tau$, the realization procedure can be simplified. We consider this case in the next section.

7 Stationary systems

In this section we specialize our results to the case of stationary systems. We show that in this case, we can construct minimal realizations by factoring rational matrices in one variable.

Definition 7.1 ([2]) *A boundary-value linear system is called stationary if its weighting pattern $w(t, \tau)$ depends only on the difference $t - \tau$.*

The following theorem characterizes the class of stationary systems.

Theorem 7.1 *The following statements are equivalent*

1- System (1-3) is stationary.

2- The canonical boundary-value operator P defined in (7) satisfies

$$O(AP - PA)R = 0 \tag{204}$$

where O and R denote respectively the observability matrix of (C, A) and the reachability matrix of (A, B).

3- There exist strictly proper rational matrices $H_c(z)$ and $H_a(z)$ such that

$$\begin{aligned} (s+\sigma)W_c(s,\sigma) &= H_c(s) - H_c(-\sigma) && (205) \\ (s+\sigma)W_a(s,\sigma) &= H_a(s) - H_a(-\sigma) && (206) \end{aligned}$$

where W_c and W_a are the transforms of the causal and the anti-causal parts of w defined in (10) and (11).

Proof We start by showing that statements 1 and 2 are equivalent. Suppose the system is stationary, i.e. $w_c(t, \tau)$ and $w_a(t, \tau)$ depend only on the difference $t - \tau$. It follows easily then that

$$\frac{\partial w_a(t,\tau)}{\partial t} = -\frac{\partial w_a(t,\tau)}{\partial \tau} \tag{207}$$

and in general for all $k, j \geq 0$

$$\frac{\partial^{k+j+1} w_a(t,\tau)}{\partial t^{k+1} \partial \tau^j} = -\frac{\partial^{k+j+1} w_a(t,\tau)}{\partial t^k \partial \tau^{j+1}}. \tag{208}$$

Evaluating (208) at $t = \tau = a$ gives

$$CA^k(AP - PA)A^j B = 0 \tag{209}$$

which clearly implies (204).

Now suppose that (204) holds. Using the Cayley-Hamilton theorem we can easily show (209) for all non-negative k and j which implies that

$$C\exp((t-a)A)P\exp(-(\tau-a)A)B = C\exp((t-\tau)A)PB \tag{210}$$

from which follows easily that

$$w(t,\tau) = \begin{cases} C\exp((t-\tau)A)(I-P)B, & a \le \tau < t \le b, \\ -C\exp((t-\tau)A)PB, & a \le t < \tau \le b, \end{cases} \tag{211}$$

and so w depends only on $t-\tau$, i.e. the system is stationary.

Now we have to show that statements 1 and 2 are equivalent to statement 3. Suppose (204) holds, then

$$\begin{aligned}(s+\sigma)W_c(s,\sigma) &= (s+\sigma)C(sI-A)^{-1}(\sigma I-A)^{-1}(I-P)B \\ &= C(sI-A)^{-1}(I-P)B - C(-\sigma I-A)^{-1}(I-P)B \qquad (212)\\ (s+\sigma)W_a(s,\sigma) &= -(s+\sigma)C(sI-A)^{-1}(\sigma I-A)^{-1}PB \\ &= -C(sI-A)^{-1}PB + C(-\sigma I-A)^{-1}PB \qquad (213)\end{aligned}$$

which implies statement 3 because we can take

$$\begin{aligned} H_c(z) &= C(zI-A)^{-1}(I-P)B \qquad (214)\\ H_a(z) &= -C(zI-A)^{-1}PB. \qquad (215)\end{aligned}$$

Suppose on the other hand that statement 3 holds and factor H_c as follows

$$H_c(z) = K(zI-F)^{-1}G. \tag{216}$$

Then it is straightforward to verify that

$$W_c(s,\sigma) = K(sI-F)^{-1}(\sigma I+A)^{-1}G \tag{217}$$

which implies that

$$w_c(t,\tau) = K\exp((t-\tau)A)G \tag{218}$$

which depends only on $t-\tau$. We can show similarly that w_a depends only on $t-\tau$ and consequently the system is stationary. □

In the stationary case, the minimality conditions (Theorem 3.1) can be simplified as follows.

Theorem 7.2 **([2])** *Stationary system (1-3) is minimal if and only if*

1- (A,Γ) is reachable where

$$\Gamma = \begin{bmatrix} (I-P)B & -PB \end{bmatrix}, \tag{219}$$

2- (Ω, A) *is observable where*

$$\Omega = \begin{bmatrix} C(I-P) \\ -CP \end{bmatrix}, \tag{220}$$

3-

$$Ker\, O \subset Im\, R, \tag{221}$$

where O and R denote respectively the observability matrix of (C, A) and the reachability matrix of (A, B).

This theorem is given for the sake of completeness; we do not use it in this paper.

It is not difficult to see that in the stationary case, i.e. when w_c and w_a depend only on $t-\tau$, $H_c(z)$ and $H_a(z)$ (defined in (214) and (215)) are the one-sided Laplace transforms of h_c and h_a respectively, i.e.

$$H_c(z) = \int_0^\infty h_c(\theta)e^{-z\theta}d\theta, \tag{222}$$
$$H_a(z) = \int_0^\infty h_a(\theta)e^{-z\theta}d\theta, \tag{223}$$

where

$$h_c(t-\tau) = w_c(t,\tau), \tag{224}$$
$$h_a(t-\tau) = w_a(t,\tau). \tag{225}$$

We can now specialize the results of the previous sections to the case of stationary systems using $H_c(z)$ and $H_a(z)$ instead of $W_c(s,\sigma)$ and $W_a(s,\sigma)$.

Theorem 7.3 *The pair of rational matrices $\{H_c(z), H_a(z)\}$ is realizable if and only if the two matrices are strictly proper.*

Proof To show necessity, suppose that $\{H_c(z), H_a(z)\}$ is realizable and let (C, P, A, B) be a realization of it. Then from (214) and (215) follows easily that $H_c(z)$ and $H_a(z)$ are strictly proper.

To show sufficiency, we suppose that $H_c(z)$ and $H_a(z)$ are strictly proper and show that $\{W_c(s,\sigma), W_a(s,\sigma)\}$, where

$$W_c(s,\sigma) = \frac{1}{(s+\sigma)}(H_c(s) - H_c(-\sigma)), \tag{226}$$
$$W_a(s,\sigma) = \frac{1}{(s+\sigma)}(H_a(s) - H_a(-\sigma)), \tag{227}$$

satisfies the conditions of Theorem 5.1 as follows. Since $H_c(z)$ and $H_a(z)$ are strictly proper, they can be factored as

$$H_c(z) = C_c(zI - A_c)^{-1}B_c, \tag{228}$$
$$H_a(z) = C_a(zI - A_a)^{-1}B_a, \tag{229}$$

which implies that

$$W_c(s,\sigma) = C_c(sI - A_c)^{-1}(\sigma I + A_c)^{-1}B_c, \quad (230)$$
$$W_a(s,\sigma) = C_a(sI - A_a)^{-1}(\sigma I + A_a)^{-1}B_a, \quad (231)$$

from which we get that $W_c(s,\sigma)$ and $W_a(s,\sigma)$ are strictly proper and have separable denominators. Thus the first condition of Theorem 5.1 is verified. The second condition can be verified by noting that

$$(s+\sigma)(W_c(s,\sigma) - W_a(s,\sigma)) = H(s) - H(-\sigma) \quad (232)$$

where

$$H(z) = H_c(z) - H_a(z). \quad (233)$$

□

Theorem 7.3 provides us with a simple test of realizability for stationary weighting patterns.

The following theorem allows us to express the dimension of minimal realizations in terms of the matrices H_c and H_a.

Theorem 7.4 *The dimension n of minimal realizations of a realizable pair $\{H_c(z), H_a(z)\}$ is given by the following equation*

$$n = d(\begin{bmatrix} H_c(z) & H_a(z) \end{bmatrix}) + d(\begin{bmatrix} H_c(z) \\ H_a(z) \end{bmatrix}) - d(H_c(z) - H_a(z)) \quad (234)$$

where $d(X)$ denotes the McMillan degree of X.

This theorem follows simply Theorem 6.1 and the following lemma:

Lemma 7.1 *Let $W(s,\sigma)$ be a strictly proper rational matrix with separable denominator and suppose that there exists a strictly proper rational matrix $H(z)$ such that*

$$(s+\sigma)W(s,\sigma) = H(s) - H(-\sigma). \quad (235)$$

Then,

$$\mu_s(W(s,\sigma)) = \mu_\sigma(W(s,\sigma)) = d(H(z)) \quad (236)$$

where $\mu_s(X)$ and $\mu_\sigma(X)$ denote respectively the s- and σ-degrees and $d(X)$ the McMillan degree of X. Moreover, if

$$W(s,\sigma) = C(sI - A)^{-1}\Gamma(\sigma) = \Omega(s)(\sigma I + E)^{-1}B \quad (237)$$

are minimal s- and σ-factorizations and

$$H(z) = K(zI - F)^{-1}G \quad (238)$$

is a minimal factorization, then there exist invertible matrices T and U such that

$$K = CT^{-1} = \Omega U^{-1}, \tag{239}$$
$$F = TAT^{-1} = UEU^{-1}, \tag{240}$$
$$G = T\Gamma = UB, \tag{241}$$

where

$$\Gamma = \lim_{\sigma\to\infty} \sigma\Gamma(\sigma) \tag{242}$$
$$\Omega = \lim_{s\to\infty} s\Omega(s). \tag{243}$$

Proof From the minimal factorization

$$H(z) = K(zI - F)^{-1}G \tag{244}$$

and (235) follows that

$$W(s,\sigma) = K(sI - F)^{-1}(\sigma I + F)^{-1}G \tag{245}$$

which implies that $(K, F, (\sigma I + F)^{-1}G)$ is an s-state-space representation of $W(s,\sigma)$. Since (F, G) is reachable, we can show that $(F, (\sigma I + F)^{-1}G)$ is also reachable. Thus (245) is a minimal s-factorization of $W(s,\sigma)$ which means that

$$\mu_s(W(s,\sigma)) = \dim F = d(H(z)). \tag{246}$$

By noting that $(K(sI - F)^{-1}, -F, G)$ is a minimal σ-state-space realization of $W(s,\sigma)$ we can similarly show that

$$\mu_\sigma(W(s,\sigma)) = \dim F = d(H(z)) \tag{247}$$

and thus we have shown that (236) holds.

$(K, F, (\sigma I + F)^{-1}G)$ and $(C, A, \Gamma(\sigma))$ are both minimal s-state-space representation of $W(s,\sigma)$ and thus thanks to the corollary of Theorem 4.2, there exists a matrix T such that

$$K = CT, \tag{248}$$
$$F = TAT^{-1}, \tag{249}$$
$$(\sigma I + F)^{-1}G = T\Gamma(\sigma). \tag{250}$$

By multiplying both sides of (250) and taking the limit as σ goes to infinity, we obtain

$$G = T\Gamma, \tag{251}$$

and thus the existence of T is proved. The existence of U can be shown similarly. □

We now give the realization procedure which can be used to find a minimal realization in terms of matrices $H_c(z)$ and $H_a(z)$. As in the general case, we start by examining the reachability and observability of minimal realizations. If n equals

$$d(\begin{bmatrix} H_c(z) \\ H_a(z) \end{bmatrix})$$

then we can deduce that minimal realizations are reachable. On the other hand, if n equals

$$d(\begin{bmatrix} H_c(z) & H_a(z) \end{bmatrix})$$

then minimal realizations are observable.

Observable case Suppose that minimal realizations are observable, i.e.

$$d(\begin{bmatrix} H_c(z) & H_a(z) \end{bmatrix}) = n. \tag{252}$$

In this case, a minimal realization can be constructed as follows:

1- perform a minimal factorization

$$\begin{bmatrix} H_c(z) & H_a(z) \end{bmatrix} = C(zI - A)^{-1} \begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix} \tag{253}$$

2- compute

$$B = \Lambda_1 - \Lambda_2 \tag{254}$$

3- find a P that satisfies

$$OPR = -\Delta \tag{255}$$

where O and R denote the observability matrix of (C, A) and the reachability matrix of (A, B), and $\Delta_{i,j}$ is the (i+j)th coefficient in the power series expansion of $H_a(z)$.

Claim *(C, P, A, B) is a minimal realization of $\{H_c, H_a\}$.*

Proof Note that since

$$\begin{bmatrix} W_c(s,\sigma) & W_a(s,\sigma) \end{bmatrix} = \frac{1}{s+\sigma}(\begin{bmatrix} H_c(s) & H_a(s) \end{bmatrix} - \begin{bmatrix} H_c(\sigma) & H_a(\sigma) \end{bmatrix}) \tag{256}$$

thanks to Lemma 7.1, C, A and B that we obtain from step 1 and step 2, are consistent with (135) and (136)

To justify step 3, simply note that P exists; it satisfies (204) and

$$H_a(z) = -C(zI - A)^{-1}PB, \tag{257}$$

from which we get

$$CA^iPA^jB = -\Delta_{i,j}, \tag{258}$$

which implies (255). □

Reachable case Suppose that minimal systems are reachable. i.e.

$$d(\begin{bmatrix} H_c(z) \\ H_a(z) \end{bmatrix}) = n. \tag{259}$$

1- perform a minimal factorization

$$\begin{bmatrix} H_c(z) \\ H_a(z) \end{bmatrix}) = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} (zI - A)^{-1} B. \tag{260}$$

2- compute

$$C = \Psi_1 - \Psi_2, \tag{261}$$

3- find a P that satisfies

$$OPR = -\Delta \tag{262}$$

where O and R denote the observability matrix of (C, A) and the reachability matrix of (A, B), and $\Delta_{i,j}$ is the (i+j)th coefficient in the power series expansion of $H_a(z)$.

Claim *(C, P, A, B) is a minimal realization of $\{H_c, H_a\}$.*

The proof is similar to the previous case.

Unreachable/unobservable case If minimal realizations are not reachable or observable, the following procedure can be used for constructing a minimal realization:

1- construct minimal factorizations

$$\begin{bmatrix} H_c(z) & H_a(z) \end{bmatrix} = \bar{C}(zI - \bar{A})^{-1} \begin{bmatrix} \Lambda_c & \Lambda_a \end{bmatrix} \tag{263}$$

$$\begin{bmatrix} H_c(z) \\ H_a(z) \end{bmatrix} = \begin{bmatrix} \Psi_c \\ \Psi_a \end{bmatrix} (zI - \tilde{A})^{-1} \tilde{B} \tag{264}$$

and let

$$\Psi = \Psi_c - \Psi_a \tag{265}$$

$$\Lambda = \Lambda_c - \Lambda_a, \tag{266}$$

2- decompose $(\bar{C}, \bar{A}, \Lambda)$ in reachable/unreachable parts and $(\Psi, \tilde{A}, \tilde{B})$ in observable/unobservable parts:

$$\bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, \tag{267}$$

and

$$\Psi = \begin{bmatrix} 0 & \tilde{C}_2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \tag{268}$$

3- construct system matrices as follows

$$A = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 V^{-1} & \star \\ 0 & \bar{A}_1 & \bar{A}_2 \\ 0 & 0 & \bar{A}_4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \bar{C}_1 & \bar{C}_2 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{B}_1 \\ \bar{B}_1 \\ 0 \end{bmatrix}, \tag{269}$$

where $\star$ indicates an arbitrary block entry and the matrix V is given by

$$V = \bar{R}\tilde{R}(\tilde{R}\tilde{R}^T)^{-1}, \tag{270}$$

where $\bar{R}$ and $\tilde{R}$ denote respectively the reachability matrices of $(\bar{A}_1, \bar{B}_1)$ and $(\tilde{A}_4, \tilde{B}_2)$.

4- find a P that satisfies

$$OPR = -\Delta \tag{271}$$

where O and R denote the observability matrix of (C, A) and the reachability matrix of (A, B), and $\Delta_{i,j}$ is the (i+j)th coefficient in the power series expansion of $H_a(z)$.

Claim (C, P, A, B) *is a minimal realization of* $\{H_c, H_a\}$.

Proof Matrices $\bar{C}$, $\bar{A}$ and Ψ obtained from (263) and (265) are consistent with (155) and (157). Similarly, matrices $\tilde{A}$, $\tilde{B}$ and Λ obtained from (264) and (266) are consistent with (156) and (158). This implies that A, B and C constructed in step 3 is consistent with (161). Thus all that we need to show is that step 4 characterizes P correctly. But we have already done this in the observable case. □

Example 7.1 Consider the stationary weighting pattern

$$w(t, \tau) = 1 \quad 0 \leq t, \tau \leq 1. \tag{272}$$

Clearly,

$$h_a(\theta) = h_c(\theta) = 1 \tag{273}$$

and

$$H_c(z) = H_a(z) = 1/z. \tag{274}$$

$\{H_c, H_a\}$ is realizable (Theorem 7.3), and the dimension of its minimal realizations is given by

$$\begin{aligned} n &= d(\begin{bmatrix} 1/z & 1/z \end{bmatrix}) + d(\begin{bmatrix} 1/z \\ 1/z \end{bmatrix}) - d(1/z - 1/z) \\ &= 1 + 1 - 0 = 2. \end{aligned} \tag{275}$$

Clearly minimal realizations are unreachable and unobservable. The realization procedure in this case is then:

1- construct minimal factorizations

$$\begin{bmatrix} 1/z & 1/z \end{bmatrix} = 1(z)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{276}$$

$$\begin{bmatrix} 1/z \\ 1/z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)^{-1} 1, \tag{277}$$

and compute

$$\Psi = 1 - 1 = 0 \tag{278}$$

$$\Lambda = 1 - 1 = 0. \tag{279}$$

2- Note that there is no reachable/observable part in this case, i.e. $\tilde{A}_1$ and $\tilde{A}_4$ have zero dimension.

3- System matrices are

$$A = \begin{bmatrix} 0 & \star \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{280}$$

4- P is any matrix satisfying

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{281}$$

i.e.,

$$P = \begin{bmatrix} \star & \star \\ -1 & \star \end{bmatrix}. \tag{282}$$

8 conclusion

In this paper we have developed a realization theory for autonomous boundary-value linear systems. In particular, we have shown that by taking 2D Laplace transforms of the causal and anti-causal parts of the candidate weighting pattern, we can examine its realizability and construct minimal realizations by factoring rational matrices in two variables having separable denominators. We have studied this factorization problem and have proposed methods for constructing such factorizations.

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